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A GUIDED TOUR OF THE WORLD OF RATIONAL EXPECTATIONS MODELS AND OPTIMAL POLICIES

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Abstract

This working paper after quickly reviewing the different types of existing macro models presents some basic tools that have proved useful for analysing monetary policy in recent years. Through the use of a simple quantitative forward-looking model of output, inflation and interest rate determination, the paper tries to familiarise the reader with some of the techniques used in research on optimal policy, including rational expectations theory, timeconsistency analysis, the Lucas critique and computer simulation techniques. The explanation proceeds gradually. First, a single linear difference equation is used to explain how solutions to models with forward-looking expectations can be obtained. Then it deals with methods used to solve more general models for optimal policies. Finally, the potential usefulness of these techniques is explained through a series of applications to monetary policy.

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1. INTRODUCTION

The purpose of this text is to present some basic tools that have proved useful for analysing monetary policy in recent years. Almost all the papers dealing with this topic now start with the same simple quantitative model of output, inflation and interest rate determination which is the economic setting for researchers such as Woodford (1999) and Clarida, Gali and Gertler (1999). It is one of a class of 'new synthesis' macroeconomic models that have as key features: intertemporal optimisation, rational expectations, imperfect competition and costly price adjustments. Nominal price rigidities create a channel through which monetary policy can affect output. This is a very useful tool, and recent research consists of variations around this model. Research by both academics and practitioners on optimal policies is now rapidly expanding; it necessitates an understanding of quite sophisticated techniques, and this paper tries to familiarise the reader with some of them. The framework used here is based on modern macroeconomic research including rational expectations theory, timeconsistency analysis, the Lucas critique and computer simulation techniques.

The layout of the paper is as follows. First, it presents a quick review of the recent history of the different types of existing models. Section two uses a very simplified model (in fact, one equation) with some desirable features in terms of expectations to explain how models with forward-looking expectations can be solved and what can be expected of a sensitivity analysis. Then section three presents the Lucas critique in that context. Afterwards, section four offers a general analysis and deals with techniques used to solve models for optimal policies. Finally, their potential usefulness is shown through a series of applications to monetary policy.

2. HISTORY OF THE VARIOUS TYPES OF MODELS

During the 1960s and 1970s, the specification, estimation, use and evaluation of large-scale econometric models for forecasting and policy analysis represented a major research topic in macroeconomics. An equation describing the behaviour of a policy instrument was incorporated into these models, facilitating model simulations of alternative policy rules. These simulations provided an estimate of the impact on the economy's dynamic behaviour of changes in the way policy was conducted. For example, a policy under which the interest rate was adjusted rapidly in response to inflation changes could be contrasted with one in which the response was more gradual.

A key hypothesis was that specification of the policy rule did not cause variations in the estimated parameters of the model. If this were not the case, then we could no longer treat the model's parameters as unchanged when altering the policy rule. The Lucas critique emphasised that the parameters of the model would shift when policy changed. The main reason underlying constant parameters was that expectations were assumed to be *adaptive* or, in other words, backward-looking and therefore unresponsive to those changes in policy that would be expected to alter expectations about future events. In other words, modelling expectations based on the past behaviour of variables or making extensive use of the usual error correction models were common practice, but can provide misleading results when the model is used to simulate fiscal and monetary policy rules.

While large scale econometric models continued to play an important role in discussions of government stabilisation policies, they fell out of fashion among academic economists as a result of the Lucas critique, the increasing emphasis on the role of expectations in theoretical models, and dissatisfaction with the empirical treatment of expectations in existing large-scale models. Subsequently, a gap emerged between applied macroeconomics as practised by academics and the macroeconomics contained in large-scale models. The academic literature showed a continued interest in small-scale rational expectations models as well as the development of larger-scale models, all of which incorporated rational expectations into some or all aspects of the model's behavioural relationships.

There is a spectrum of modelling approaches which vary according to the degree of theoretical structure as opposed to data-mining.

The Lucas critique implies a constructive way of improving on conventional evaluation techniques by modelling economic phenomena in terms of structural parameters. By structural we simply mean invariant with respect to policy intervention. It is necessary to rely more heavily on economic theory here. This favours use of *Dynamic General Equilibrium Models* to analyse the effects of alternative feedback rules for monetary policy, since these models stand up better to the Lucas critique. These models are derived entirely from optimising behaviour by economic agents and represent the beginning of the spectrum. Originally, in line with Kydland and Prescott (1982), dynamic general equilibrium models were calibrated, while nowadays many of them are at least partially estimated. The basic idea of calibration is to choose parameter values on the basis of microeconomic evidence, and then to compare the model's predictions concerning the variances and covariances of various series with those in the data. From Romer (1996, p. 180): "Calibration has two potential advantages over estimating models econometrically. First, because parameter values are selected on the basis of microeconomic evidence, a large body of information beyond that usually employed can be brought to bear, and the models can therefore be held to a higher standard. Second, the economic importance of a statistical rejection, or lack of rejection, of a model is often hard to interpret. A model that fits the data well along every dimension except one which is unimportant may be overwhelmingly rejected statistically. Or a model may fail to be rejected simply because the data are consistent with a wide range of possibilities". However, not all the parameter values can be pinned down by microeconomic evidence. Such a tool has also been built at the NBB these last few years, Dombrecht and Wouters (2000).

At the other end of the spectrum, *vector autoregression models* (VARs) estimate statistically the dynamic interactions between a set of variables without imposing strong theoretical assumptions. So VARs capture average past experience in a less restricted way. However, since VARs are reduced form models they are especially vulnerable to structural economic changes and to the Lucas critique. Moreover, *unrestricted* VARs yield no economic interpretation of average past experience. Structural vector autoregressions (SVARs) try to improve this situation by introducing "identifying" restrictions. Actually, the latter are of two kinds : either restrictions on the matrix that links the observable VAR residuals to the underlying structural disturbances (e.g. monetary policy shock affects output with a lag), or restrictions on the long-run effects of the disturbances on observed variables (e.g. long-run neutrality of money implies that a monetary policy shock has no permanent effect on output). These restrictions enable researchers to assign an economic interpretation to each of the disturbances in the model. The advantage of SVARs is that they can be used to diagnose the sources of shocks that have affected links between variables. Observed patterns of past behaviour can be interpreted as system responses to particular kinds of economic shocks. The vast literature on VARs initiated by Sims (1980) also provided a useful benchmark against which structural models could be gauged. A criticism of (S)VARs is that this approach misses important information available to policy makers. In particular, many of the VAR models used to assess monetary policy fail to incorporate forward-looking variables.

However, for forecasting and day-to-day sensitivity analysis we often need to compromise between theoretical structure and data mining, but the preceding discussion reveals that a clear and delineate treatment of expectations is a minimum requirement. "Larger-scale econometric models have proven useful to central banks in providing answers to questions related to the design and implementation of monetary policy, and within the last few years, a new generation of large-scale econometric policy models have come into use. ... . These econometric models are designed to address specific questions of relevance for the actual design of monetary policy. The FRB/US model is structured to allow simulations to be conducted under alternative assumptions about expectations formation. ... . Other countries have also actively developed econometric models for policy work combining both estimated and calibrated relationships." (Walsh 1998 p. 34) Walsh reports models for Canada and New Zealand, but the EEC’s Quest model and the IMF’s Multimod can also be mentioned. Recently, we developed at the NBB a model in the spirit of the FRB/US model, Jeanfils (2000). This approach has many advantages. First it allows us to use a forward-looking approach which, as monetary policy takes time to affect output and inflation, is essential to monetary policy-making. Its flexibility allows us to examine the model’s sensitivity to alternative expectation scenarios. Second, the introduction of the rational expectations hypothesis, which explicitly distinguishes between lags stemming from either the economic environment (adjustment costs and inertia effects) or expectations, limits the susceptibility of the model to the Lucas critique.

The rational expectations hypothesis is now so widespread in the macroeconomic literature that it seems undoubtedly a valid approximation of the way expectations are formed, and there is now no point in working without it. The assumption was introduced in somewhat simplified Neo-Classical models, but there is currently a pervasive treatment of expectations in a large number of other models basically structured along New-Keynesian lines. The rational expectations hypothesis should indeed be separated from the market-clearing hypothesis. The former simply assumes that agents form their expectations in an informed and efficient manner (they do not make *systematic* errors). Expectations are essentially the same as the predictions of the relevant economic theory. If that theory is a Keynesian one in which market disequilibrium can persist, then rational expectations may be seen as a valid Keynesian mechanism.

Moreover, many macroeconomic events have shown some kind of jumpiness, in the rational expectations sense, which can hardly be explained and modelled without forward looking expectations. Asset valuation is the most obvious example: the current price of an asset is determined by its expected future incomes. The importance of forward-looking expectations is also true for variables that cannot be freely and quickly adjusted. The presence of adjustment costs means that agents must balance the cost of deviation from their desired level for the variables in question with the cost of adjusting to reach that desired level. In this case, the importance of expectations is determined by the strength of constraints on dynamic adjustment: if a variable is slow to respond (e.g. once a firm's optimal level of capacity has been decided, the decision to invest also takes time to generate productive capital), forecasts about more distant economic conditions are needed (the greater the friction, the farther into the future expectations must go) to find a route for adjustment towards the target level.

3. A SIMPLE, ONE- EQUATION FORWARD-LOOKING MODEL

Most of the questions that economists have to answer involve expectations of future interest rates, exchange rates, oil prices and so on. A reasonably sophisticated treatment of expectations is thus essential. However the explicit introduction of forward-looking expectations formation in macro-models gives rise to several complications.

It is easier to outline the role of expectations using a simple model for which a solution can be calculated analytically. To keep the maths as simple as possible, I restrict the model to a linear relationship between one variable, one expectation and one stochastic shock.

This simple model assumes that inflation depends on the expected next period's inflation plus an exogenous shock

$$\pi_t = \beta E_t \pi_{t+1} + \delta z_t \tag{1}$$

where π_t is the rate of inflation at time t , β is a discount factor and $E_t\pi_{t+1}$ is the expected inflation rate between period t and period t+1. The variable z is an exogenous shift variable which could represent a true exogenous variable for the model -e.g. oil price- or a policy variable -e.g. interest rate or money supply. It could also represent a stochastic error term as in an econometric equation and, in that case, δ=1. If z_t is a policy variable, then we can represent the design of alternative policy rules by specifying a different stochastic process for z_t. All variables are expressed as deviations from their long-run or target level. Section five gives a structural interpretation of this equation.

Suppose the question we have to answer is about the effect of a shift in z_t on inflation.

The next period's inflation will be determined in the same way as this period's inflation:

$$\pi_{t+1} = \beta E_{t+1} \pi_{t+2} + \delta z_{t+1} \tag{2}$$

Note that all future rates of inflation can be expressed in the same way. By iterating forward, we finally arrive at an expression for current inflation in terms of current and expected future shocks, the weights of which decrease with time :

$$\pi_t = \delta\left[z_t + \beta E z_{t+1} + \beta^2 E z_{t+2} + \beta^3 E z_{t+3} + \ldots\right] \tag{3}$$

Therefore, to examine a shock in variable z, it is necessary to specify all its future values *even beyond the sample* we are interested in. For example, in order to examine the sensitivity of a two-year forecast in a quarterly model (t=1, …, 8) to a change in money supply, it is also necessary to make assumptions about future values of money for t=9,…. Does it remain at the same value as in t=8; or does it go back to the baseline, and in the latter case does it return gradually or in one jump ?

In accordance with Taylor (1993a), I now explain how these questions can affect the results. Suppose that z_t has the very general form

$$z_t = \sum_{i=0}^{\infty} \theta_i \varepsilon_{t-i} , \qquad (4)$$

where θ_i are parameters and ε_i is a serially uncorrelated innovation with zero mean.

This is a very general representation. Indeed, consider for instance the particular case of an AR(1) process

$$z_t = \mu z_{t-1} + \varepsilon_t , \mu \leq |1|.$$

Solving backwards gives

$$z_t = \sum_{i=0}^{\infty} \mu^i \varepsilon_{t-i}$$

which means that the weights on ε must be decreasing geometrically.

The simplified model used to illustrate the impact of shocks consists of equations (1) and (4). The following distinctions as to the nature of a particular shock can be made:

(i) Temporary versus permanent shocks

- z_t is purely temporary when θ_0=1 and θ_i=0, for i>0. Any shock z_t is expected to disappear (or to return to the baseline) immediately after the period in which it has occurred:
  $E_t z_{t+i} = 0$ for i>0 whatever the actual value of z_t;

- z_t is permanent when θ_i=1, $\forall i \geq 0$. Any shock z_t is expected to remain forever. Then all future values of z_{t+i} are expected to be equal to z_t : $E_t z_{t+i} = z_t$ for i>0 whatever the actual value of z_t;

- in between the latter two extremes, z_t exhibits intermediate persistence. With $\theta_i = \mu^i$, a wide range of intermediate persistence processes can be modelled by letting μ vary from 0 to 1 (the larger the value of μ, the longer the persistence). In this case the shock dies out geometrically in accordance with an AR(1) process.

(ii) Anticipated versus unanticipated shocks

In forward-looking expectations models, the response also depends on whether the shift in the variable is (credibly in the case of a policy variable) anticipated or unanticipated. If we have to analyse now (in i=0) the effect of a shock that will occur k periods ahead, e.g. an interest rate increase next quarter, we can simulate:

- an unanticipated shift by setting $\theta_i=0$ for i=0 to k-1, i.e. from now up to the period preceding the shift, so that $E_i(z_k)=0$ for $0\leq i<k$ and $E_i(z_k)=\varepsilon_k$ for $i\geq k$;

- an anticipated shift by making the expectation of the variable z_k made at any time equal to z_k . Such a shock anticipated k periods in advance will be represented by $z_t=\varepsilon_{t-k}$ and, if it is expected to die out geometrically, will be simulated by setting $\theta_i=0$ for i=0, … , k-1 and $\theta_i=\mu^{i-k}$ for i=k,k+1, … $0<\mu<1$.

Combining these definitions, four cases of one-period shocks are now examined. The intention is to find a timeframe for inflation, π_t, given the expected characteristics of z_t. In order to obtain the results that follow, we can use the method of undetermined coefficients. It consists in representing π_t in an unrestricted infinite moving average form

$$\pi_t = \sum_{i=0}^{\infty} \gamma_i \varepsilon_{t-i} . \tag{5}$$

Then the solution for π_t requires finding the values for the undetermined coefficients γ_i such that equations (1) and (5) are satisfied. These coefficients are obtained by substituting π_t and $E_t\,\pi_{t+1}$ in (1) making use of (5) and then solving γ_i in terms of β, δ and μ.

Results are also depicted in figures 1-(a) to (d).

(a) Unexpected and temporary

$$\pi_t = \delta\varepsilon_t$$

The interpretation is trivial since inflation only changes the period in which the exogenous variable, e.g. money supply, is shifted, and is unaffected otherwise. This is due to the fact that it is expected to return to its original level (of zero) the period after.

(b) Unexpected and dying out gradually

$$\pi_t = \frac{\delta}{(1-\beta\mu)} \sum_{i=0}^{\infty} \mu^i \varepsilon_{t-i}$$

$$\pi_t = \frac{\delta}{(1-\beta\mu)} z_t$$

For $\mu < 1$, inflation increases by less than the initial impulse in the money supply because of the expected deflation that occurs as inflation returns to its base value (normalised at zero in the figure). In the case of a permanent shock ($\mu=1$), inflation would move once and for all proportionally to the shock with a coefficient of $\frac{\delta}{(1-\beta)}$ higher than in the case $\mu < 1$ because there would be no expected future deflation, since the shock remains at its new level.

(c) Anticipated and temporary

Here the shock is anticipated k periods in advance : $z_t=\varepsilon_{t-k}$, $\theta_k=1$, $\theta_i=0$ for $i\neq k$

$$\pi_t = \delta\left[\beta^k \varepsilon_t + \beta^{k-1}\varepsilon_{t-1} + \ldots + \beta\varepsilon_{t-(k-1)} + \varepsilon_{t-k}\right]$$

$$\pi_t = \delta\left[\beta^k z_{t+k} + \beta^{k-1} z_{t+k-1} + \ldots + \beta z_{t+1} + z_t\right]$$

In this case inflation "jumps" at the time of the announcement and then gradually rises until the increase in the money supply actually happens. Note that at the actual date of the shock, the increase in inflation is the same as in the unexpected case.

(d) Anticipated and dying out geometrically

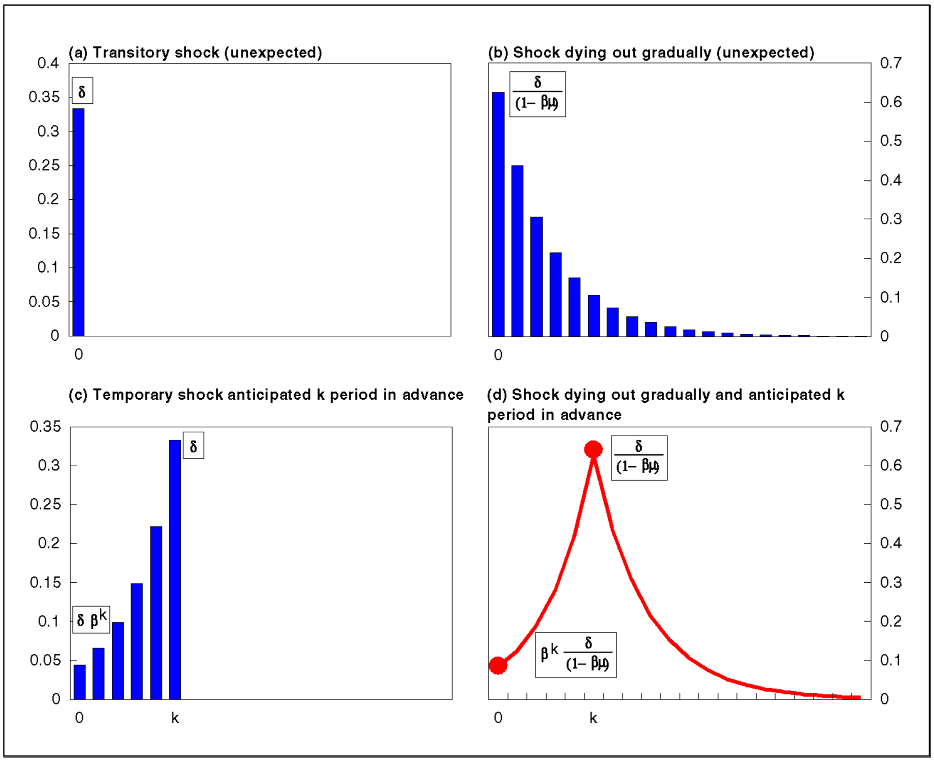
Here the shock is anticipated k periods in advance : $\theta_i=\mu^{i-k}$, $z_t = \sum_{i=0}^{\infty}\mu^i\varepsilon_{t-i}$ for $i\geq k$, $\theta_i=0$ for $i<k$)

$$\pi_t = \frac{\delta}{1-\beta\mu}\left[\begin{array}{l}\beta^k\varepsilon_t+\beta^{k-1}\varepsilon_{t-1}+\ldots+\beta\varepsilon_{t-(k-1)}+\varepsilon_{t-k}\\ +\mu\varepsilon_{t-(k+1)}+\mu^2\varepsilon_{t-(k+2)}+\ldots\end{array}\right]$$

$$\pi_t = \frac{\delta}{1-\beta\mu}\left[\begin{array}{l}\beta^k z_{t+k}+\beta^{k-1}z_{t+k-1}+\ldots+\beta z_{t+1}+z_t\\ +\mu z_{t-1}+\mu^2 z_{t-2}+\ldots\end{array}\right]$$

In this case, too, the anticipation of an increase in the money supply induces a jump in inflation which then increases gradually until the shock actually takes place.

Figure 1 - Inflation: impulse response to various configurations of shocks



These small examples show the importance of adequately specifying the timing and nature of the shocks.

4. THE LUCAS CRITIQUE

Suppose that a policy advisor knows that inflation is determined by

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t + \varepsilon_{\pi t} \tag{6}$$

in which y_t represents the output gap: the deviations of the log of real output from its trend path.

Here there are two shocks to the system: a demand shock (the output gap) y_t and a supply shock $\varepsilon_{\pi t}$. Suppose that

$$\varepsilon_{\pi t+1} = \tau_\pi \varepsilon_{\pi t} + \upsilon_{\pi t+1} \tag{7}$$

where υ_π is a white noise disturbance.

For simplicity, suppose that the monetary authorities can perfectly control the output gap through their instrument, and that their objective is to stabilise output so that the output gap remains fixed at zero. The policy rule is thus $y_t = 0$ (zero output gap policy). But suppose also that, under this policy, inflation is judged too volatile. The policy maker therefore calculates how y_t can be manipulated to reduce inflation volatility.

Under the zero output gap policy combined with the law of motion of ε_π in (7), the observed behaviour of inflation in the past was

$$\pi_t = \tau_\pi \pi_{t-1} + \frac{\upsilon_{\pi t}}{(1-\beta\tau_\pi)} \tag{8}$$

Conventional policy evaluation might proceed as follows. In a first step, estimate τ_π in the reduced-form equation (8) over the sample period. Second, use this estimated equation as a model generating expectations i.e. $E_t\pi_{t+1} = \tau_\pi\pi_t$. Finally, substitute these expectations into (6) to yield

$$\pi_t = \beta\tau_\pi\pi_t + \kappa y_t + \varepsilon_{\pi t} \tag{9}$$

so that the conventional model of inflation becomes:

$$\pi_t = \frac{\kappa y_t + \varepsilon_{\pi t}}{(1-\beta\tau_\pi)} \tag{10}$$

This type of relation was often estimated without reference to a structural model. Then an advisor could be asked to answer a typical question of the kind "Can you estimate the influence of the output gap on inflation ? " The policy maker asking this question knows that if the econometrician finds a coefficient of one, for example, he will be able to alter inflation by one percent by changing the output gap by the same percentage. But if he used to pursue a zero output policy, this conclusion would obviously be erroneous.

Considering a new policy rule of the form

$$y_t = g\varepsilon_{\pi t-1} \tag{11}$$

one would be tempted to substitute (11) in (10) and to use the result to calculate the optimal g. This procedure is false if expectations are rational since the coefficient relating output gap to inflation in (10) is not structural. The true coefficient is a function of the policy rule. Actually, (10) is only valid under the policy rule $y_t = 0$.

The correct approach would have been to substitute $y_t = g\varepsilon_{\pi t-1}$ directly into (6) and then to calculate the stochastic process for π_t. This results in the following MA(1) process:

$$\pi_t = \frac{1+\beta\kappa g}{1-\beta\tau_\pi}\varepsilon_{\pi t} + \kappa g \varepsilon_{\pi t-1} \tag{12}$$

This relation which differs from (10) in which the policy rule has been substituted, can be used to simulate alternative policy rules. For instance, if the monetary authorities mainly care about inflation volatility, the optimal rule can be found by minimising the variance of inflation with respect to g.

5. LINEAR MODEL WITH MORE THAN ONE VARIABLE

Since economic policies are discussed in a framework that involves more than one equation, the simple model above is generalised. In this respect, it is useful to recall that the monetary policy literature has mainly followed two routes. First, following on from Taylor (1993b), many authors have advocated simple policy rules that receive a given functional form based on both economic and control theory. Coefficients of such rules are chosen either by reference to historical experience, as in the Taylor rule, or by optimisation in which the discounted value of expected inflation deviations is minimised by the choice of coefficients given the structure of the economy. In the latter case, the rule is called an 'optimal' simple rule. In order to gauge the performance of the rules, many papers analyse the behaviour of a given model using different policy rules by modifying their parameters or their arguments, using rule-consistent inflation forecasts rather than current inflation, Batini and Haldane (1999), or by changing the forecast horizon, etc. Second, more recently following the impetus given by Currie and Levine (1993), the dominant approach consists in specifying an objective function for the policy maker and then solving a given model to obtain the optimal policy. This second route uses optimisation not just to choose the coefficients of a rule but to set the entire trajectory for interest rates that minimises the representative loss function.

To describe the solution methods and to introduce optimal policies, I use a simple model called the 'new synthesis' model ascribed to Woodford (1999) and Clarida,Gali,Gertler (1999). The model contains two log-linear approximations to Euler equations describing decisions of households and firms: a dynamic "IS" curve relating the output gap inversely to the real interest rate and an expectational Phillips curve relating inflation positively to the output gap already given in (6). They are written as:

$$y_t = -\varphi[i_t - E_t\pi_{t+1}] + E_t y_{t+1} + \varepsilon_{yt} \tag{13}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t + \varepsilon_{\pi t} \tag{14}$$

where i_t is the deviation of the nominal interest rate (the central bank instrument) from its steady state value.

Since ε_{yt} shifts the IS curve, it is interpreted as a *demand shock*, e.g. variations in government purchases, in consumer confidence, etc. The negative effect of the real rate on current output reflects intertemporal substitution in consumption. The second equation stems from price-setting behaviour à la Calvo (1983) in which, due to assumptions about preferences, technology and the labour market as demonstrated in Gali and Gertler (1999), the marginal cost has been replaced by a linear function of the output gap. Calvo's approach makes it possible to rationalise price rigidities. The term $\varepsilon_{\pi t}$ shifts the price equation and is thus a price shock. It will be called a *cost-push* shock in the sense that it captures anything that affects marginal costs other than demand shocks, e.g. changes in distortionary taxation, in degree of market power of firms, in the wage premium over marginal productivity, etc. This term allows the model to generate variations in inflation arising independently of movements in excess demand, y_t.

Note that it is interesting to iterate (14) forward to obtain

$$\pi_t = E_t \sum_{i=0}^{\infty} \beta^i \left[\lambda y_{t+i} + \varepsilon_{\pi t+i}\right] \tag{15}$$

which highlights the sources of inflation : the present value of excess demand , y, and the present value of cost- push variables.

In the presence of nominal rigidities, monetary policy can change the short-term real interest rate by modifying the nominal rate and in turn affecting output. Iterating forward (13) to yield:

$$y_t = E_t \sum_{i=0}^{\infty} \left[-\varphi(i_{t+i} - \pi_{t+1+i}) + \varepsilon_{yt+i}\right] \tag{16}$$

shows that the output gap depends on the expected future pattern of real rates and demand shocks. In this forward-looking context, expectations about the way the central bank will set interest rates in the future play a crucial role.

5.1. A simple rule without specifying an objective function

If i_t is taken as the instrument, it is not necessary to specify a money market equilibrium condition (LM curve). However the model (13)-(14) would be unstable if the nominal interest rate were to remain fixed. In order to avoid divergence between aggregate demand and aggregate supply it is necessary to introduce a feedback rule for the interest rate.

Suppose the central bank uses the following ad hoc policy rule, known to the public:

$$i_t = \gamma_\pi E_t \pi_{t+1} + \gamma_y y_t \tag{17}$$

Modified versions of this rule have been successfully estimated by Clarida, Gali and Gertler (1997) for the US, Germany and Japan. It is a forward-looking version of the Taylor rule in which the central bank responds to expected inflation, rather than to lagged inflation.

Substituting the reaction function (17) in (13) results in a system of two linear difference equations that can be solved for the movement in output and inflation when shocks hit the economy. Note that the policy rule is substituted from the start and not in the solution, so that this strategy is Lucas-critique-proof. The solution of the model can be represented as a two dimensional linear vector autoregressive moving average system with cross-equation constraints:

$$\pi_t = \theta_{\pi\pi}(L)\varepsilon_{\pi t} + \theta_{\pi y}(L)\varepsilon_{yt}$$

$$y_t = \theta_{y\pi}(L)\varepsilon_{\pi t} + \theta_{yy}(L)\varepsilon_{yt}$$

where $\theta_{i,j}(L)$ are polynomials in the lag operator and their parameters are complex functions of the structural parameter of the model, κ, β, φ, γ_π, γ_y.

5.2. Optimal policies

The current literature emphasises the difference between simple rules and fully optimal solutions derived from an intertemporal optimisation procedure. To follow the second route, we need an objective function for the policy maker. According to the literature, the objective of monetary policy is to minimise the squared deviations of output and inflation (its target variables) from their respective target levels. Sometimes the variability of the interest rate is also minimised. In this context the objective function may be written as[1]:

$$\text{Min } W_0 = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t L_t \right\} \tag{18}$$

$$L_t = \{(\pi_t - \pi^*)^2 + \lambda_y (y_t - y^*)^2 + \lambda_i (i_t - i_{t-1})^2\} \tag{19}$$

s.t. the structure of the economy given in (13) and (14)

Suppose in addition that the shocks ε_{yt} and $\varepsilon_{\pi t}$ evolve according to

$$\varepsilon_{\pi t+1} = \tau_\pi \varepsilon_{\pi t} + \upsilon_{\pi t+1} \tag{20}$$

$$\varepsilon_{yt+1} = \tau_y \varepsilon_{yt} + \upsilon_{yt+1} \tag{21}$$

where $0 \le \tau_\pi, \tau_\lambda \le 1$ and $\upsilon_{\pi t}$, υ_{yt} are white noise processes with variances σ_π^2 and σ_y^2.

This simple model, like many other macro models, especially those used to analyse monetary policy, involves forward-looking behaviour. This forward-looking characteristic exacerbates the differences between possible optimisation procedures. In

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[1] The relative weights of the targets do not come from a utility based welfare function and are therefore not entirely suited for analysis of optimal policy in terms of a welfare criterion, but an extension along these lines is not the purpose of the present paper. On the other hand, this approach has been justified by Rotemberg and Woodford (1998) and Woodford (1999a) as a quadratic approximation of the theoretically correct welfare measure, the expected utility level of the representative household. In this case, the relative weights depend on the deep preference parameters of the model.

addition, except in very simple models, no analytical solution exists, implying that some numerical methods are needed to find the optimal policy and the rational expectations equilibrium. I therefore derive the equilibrium first, and then present equilibrium responses to shocks for some small stylised models.

Most of these models can be cast in a more general linear-quadratic optimal control framework as described in Currie and Levine (1993), Söderlind (1999) or Hansen and Sargent (2000), enabling us to deal with more sophisticated economies. This framework provides the most frequently used tools in research on optimal policies.

Consider the linear model

$$\begin{bmatrix} X_{t+1} \\ E_t \chi_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ \chi_t \end{bmatrix} + Bu_t + \begin{bmatrix} \upsilon_{t+1} \\ 0 \end{bmatrix} \tag{22}$$

where X_t is a n_1 x 1 vector of predetermined (backward-looking) variables that can be lagged endogenous or exogenous, X_0 is given , χ_t is a n_2 x 1 vector of forward-looking variables which are free to "jump" in response to news, u_t is a k x 1 vector of instruments. For example, in the simple model above the elements of X_t are the exogenous shocks ε_{yt} and $\varepsilon_{\pi t}$, and those of χ_t are π_t and y_t , while the only instrument is the short-term interest rate. At the beginning of period t, X_t and υ_t are realised . Then u_t is set by the central bank. Finally χ_t results and period t ends. All variables are observable.

Defining the vector $Y_t \equiv [X_t, \chi_t]$, the objective function can be compactly written as

$$W_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{bmatrix} Y_t' & u_t' \end{bmatrix} \begin{bmatrix} Q & U \\ U' & R \end{bmatrix} \begin{bmatrix} Y_t \\ u_t \end{bmatrix} \right\} \tag{23}$$

or

$$W_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ Y_t' Q Y_t + 2 Y_t' U u_t + u_t' R u_t \right\} \tag{23'}$$

From (23), it is clear that, given the simple structure of the economy as described in (13), (14) and (18) to (21), Q is a diagonal matrix with zeros and the weights attached to each target on the main diagonal, and since there is no cross-product term between the states and the control, U is a vector of zero. If the central bank does not care about interest rate fluctuations, i.e. $\lambda_i = 0$ in (19), R is a vector of zero.

The policy problem is to choose a sequence for the instruments u_t in order to generate a timeframe for the target variables that minimises the loss function (23) subject to the structure of the economy (22). The combination of quadratic loss and linear constraints yields a certainty-equivalent decision rule for the path of the instruments. This means that the optimal decision rule in this case is identical with the rule for the corresponding non-stochastic problem. In other words, although the objective function depends on the variance-covariance of the shocks through (22), the optimal decision is independent of this variance-covariance. This is particularly useful to a policy-maker because it implies that a rule appropriate to all initial states of the system and to all types of disturbance is available. The different optimisation procedures are summarised below, in accordance with, Currie and Levine (1993), Söderlind (1999) and Woodford (1999b).

The rest of this section describes the three optimisation procedures that are usually followed in the literature: commitment, discretion and optimal simple rules.

5.2.1. (Unrestricted) commitment equilibrium

In this case, the policy maker is assumed to be able to commit himself at time 0, once and for all, to a reaction that minimises his loss function (23) subject to the structure of the economy (22). He can therefore give a credible signal that the reaction will be sustained over time. The possibility of commitment matters in this kind of optimisation: because of the presence of the forward terms $E_t y_{t+1}$ and $E_t \pi_{t+1}$ in equation (13) and (14), the value of the *period* loss function L_t that can be achieved at a given moment depends upon the private sector expectations about the future values of the endogenous variables and consequently about the way the policy maker can affect those expectations.

To solve the problem it is useful to write a Lagrangian of the form:

$$\ell_0 = \underset{\{u_t\}}{\mathrm{Min}}\, E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[Y_t' Q Y_t + 2Y_t' U u_t + u_t' R u_t + 2\Xi_{t+1}' \left(AY_t + Bu_t + \xi_{t+1} - Y_{t+1} \right) \right] \right\} \quad (24)$$

where $\xi_{t+1} = \left[\upsilon_{t+1}, \chi_{t+1} - E\chi_{t+1}\right]$.

An optimal plan, i.e. an equilibrium, is a set of (bounded) processes[2] $\left\{X_t, \chi_t, u_t, \Xi_t\right\}_{t=0}^{\infty}$ that must satisfy:

(i) the n+k first-order conditions with Ξ_{t+1} and Ξ_t are obtained by differentiating the Lagrangian with respect to u_t and to Y_t:

$$u_t = -R^{-1}\left[U' Y_t + B' E_t \Xi_{t+1}\right], \text{ if } R^{-1} \text{ exists}^3 \tag{25}$$

$$\beta A' E_t \Xi_{t+1} = \Xi_t - \beta Q Y_t - \beta U u_t; \tag{26}$$

and this last condition can be more compactly written if we use (25) to eliminate the instruments as:

$$\beta\left(A' - UR^{-1}B'\right)E_t \Xi_{t+1} = \Xi_t + \beta(UR^{-1}U' - Q)Y_t \tag{27}$$

(ii) the structure of the model (22)

(iii) the n_1 initial conditions from the predetermined X_0 and the n_2 initial conditions from $\Xi_{20} = 0_{(n2x1)}$, where $\Xi_t' = \left[\Xi_{1t}' \quad \Xi_{2t}'\right]$ is partitioned so that Ξ_{1t} is of dimension n_1 associated with predetermined variables and Ξ_{2t} is of dimension n_2 associated with the forward-looking variables[4]. This partitioning is useful since the Lagrange multipliers associated with the non-predetermined variables are themselves predetermined, whilst the Lagrange multipliers associated with predetermined variables are non-predetermined[5].

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[2] The corresponding sequence in the simple model is $\{\varepsilon_{\pi t}, \varepsilon_{yt}, \pi_t, y_t, i_t, \Xi_{ISt}, \Xi_{ASt}\}$ where Ξ_{ISt} and Ξ_{ASt} are the Lagrange multipliers associated with respectively (13) and (14).

[3] The procedure can be adapted if R^{-1} does not exist.

[4] Currie and Levine (1993) p. 153 show that a condition for W_0 to be optimal is $\Xi_{20} = 0$.

[5] See p. 102 in Currie and Levine (1993).

Leaving aside technical details concerning its derivation, which are described in Söderlind (1999), we can show that, in the commitment solution, the predetermined variables can be expressed in terms of their own lagged values and the shocks:

$$\begin{bmatrix} X_{t+1} \\ \Xi_{2t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ \Xi_{2t} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0_{(n2x1)} \end{bmatrix} \tag{28}$$

and the non predetermined variables (Ξ_{1t} , u_t , χ_t) are written as linear functions of the predetermined. For instance, instruments are given by the second row in

$$\begin{bmatrix} \Xi_{1t} \\ u_t \\ \chi_t \end{bmatrix} = -F^{Commit} \begin{bmatrix} X_t \\ \Xi_{2t} \end{bmatrix} \tag{29}$$

If R is invertible then (25) gives the optimal policy: $Y_t \equiv [X_t, \chi_t]$ is given by (28) and (29) and $E_t \Xi_{t+1}$ can be calculated by the same equations. If we partition A according to

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

note that, given the initial values $\Xi_{20}=0_{n2x1}$ and since there is no shock in Ξ_{2t} , (28) implies that

$$\Xi_{2t+1} = A_{21} X_t + A_{22} \Xi_{2t}$$

and this last condition can be solved backwards to give Ξ_{2t} as a function of the lags of X_t :

$$\Xi_{2t} = A_{21} \sum_{s=0}^{t} A_{21}^{t-s-1} X_s$$

Substituting this in (29), it is possible to rewrite the optimal rule under commitment as a discounted sum of $\{X_s\}_{s=0}^{t}$ only. This representation involves many lags (= a rule with memory), making the system dynamic, as we will demonstrate in some illustrative examples.

Anticipating a little, it is worth noting that in the commitment case:

- contrary to the discretion case, the central bank does not take private sector expectations as given, but recognises that its policy choice effectively determines those expectations.
- contrary to the simple rule case, the choice of rule is not restricted to being dependent on the contemporaneous value of the shock, but instead is allowed to be a function of the entire history of shocks.

In the commitment case the optimal policy also depends on the shadow price of the forward-looking variables. This policy is only optimal ex-ante, but ex-post it becomes sub-optimal and there is an incentive to renege. Indeed, a once-and-for-all commitment made today about the way the central bank will adjust its policy to affect endogenous variables in latter periods would not necessarily coincide with what would be optimally chosen in latter periods without such an advance commitment. This is the problem of time-inconsistency (see Kydland and Prescott (1977) or Barro and Gordon (1983)). Time-inconsistency stems from the presence of Ξ_{20} . Actually, the decision maker who will have to solve the problem at a later date T will choose $\left\{Y_{T+i}, u_{T+i}, \Xi_{T+i}\right\}_{i=0}^{\infty}$ that satisfy:

(i) first-order conditions with Ξ_{T+1} and Ξ_T ;

(ii) the structure of the model (22);

(iii) and the initial conditions $\Xi_{2T} = 0$ which are in general <u>not</u> satisfied by the optimal plan under commitment chosen <u>at date zero</u>, the time of the initial optimisation.

There is however another equilibrium concept involving another type of commitment that Woodford (1999c) and McCallum and Nelson (2000) find more attractive. Instead of using (27) and (22) with the initial conditions (iii) to determine the paths of $\left\{Y_{T+i}, u_{T+i}, \Xi_{T+i}\right\}_{i=0}^{\infty}$, the central bank can use (27) and (22) without any initial conditions by applying (27) in all periods. This "*timeless perspective*" means ignoring any conditions prevailing at the start of the regime; e.g. by seeing the decision to apply (27) as being made far in the past. There is no dynamic inconsistency in this case: the values

$\{Y_2, u_2, \Xi_2\}$ chosen by this procedure in period 2 agree with the values chosen expectationally in period 1[6]. In other words, the central bank adopts "not the pattern of behavior from now on that it would be optimal to choose, taking previous expectations as given, but rather the pattern of behavior to which it would have wished to commit itself at a date far in the past, contingent upon the random events that have occurred in the meantime. This timeless perspective ensures that the program of action that one would choose at date one is indeed the continuation of the program that one would choose at date zero: in each case it is the program that one would have wished to commit to at a date far in the past." (Woodford (1999c) p. 18). Actually, many studies of optimal policy in forward-looking models, including Clarida, Gali and Gertler (1999), have considered policies which are labelled "commitment" but which should be considered as timeless perspective policies, since these policies ignore the period 0 conditions and use only the remaining portion of the commitment conditions.

5.2.2. Discretion equilibrium

Since he cannot credibly manipulate beliefs in the absence of commitment, the policy maker takes the expectations of private agents as of time t as given (Nash equilibrium). Then, depending on the central bank's optimal rule, the private sector forms beliefs rationally. The private sector forms its expectations by taking account of how the central bank adjusts its policy, given that it is free to re-optimise every period. In a rational expectations equilibrium, the central bank has no incentive to change its plan unexpectedly even though it has the discretion to do so. The policy is therefore time-consistent. The optimal policy problem at each date can be cast in a dynamic-programming format, and the solution results in a policy reaction as a function of the current state variables only.

Such a time-consistent solution must satisfy Bellman's "Principle of Optimality" which states that an optimal policy has the property that, whatever the initial state and decision, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. Here it is useful to rewrite the loss function (23') as

$$W_t = \left\{ Y_t^{'} Q Y_t + 2 Y_t^{'} U u_t + u_t^{'} R u_t + \beta E_t W_{t+1} \right\} \tag{30}$$

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[6] For a clear exposition of this procedure see McCallum and Nelson (2000).

The dynamic programming solution then seeks a stationary solution in which W_t is minimised at time t subject to (22), in the knowledge that a similar procedure will be used to minimise W_{t+1} at time t+1. In other words, the Principle of Optimality says that the optimal value of W_t at time t, say f*(X(t)), must satisfy the Bellman equation

$$f^*(X_t) = \underset{u_t}{\mathrm{Min}}\left\{Y_t'QY_t + 2Y_t'Uu_t + u_t'Ru_t + \beta E_t f^*(X_{t+1})\right\} \tag{31}$$

The optimal response in the once-and-for-all commitment equilibrium was found to be a linear function of X_t and Ξ_{2t}. The presence of Ξ_{2t} which resulted in a representation involving lags of X_t was the source of the time inconsistency in that case. This reveals a necessary condition for a time-consistent rule, namely that it must not depend on past values of the state vector. Therefore, a solution should be of the form:

$$u_t = -F^{disc}X_t \tag{32}$$

To obtain an optimal policy linear in X_t the loss function must be quadratic in X_t only. Then we write $W_{t+1} = X_{t+1}'V_{t+1}X_{t+1}$ in (31). The complication of this optimisation is that the objective function depends on the forward-looking variables which are endogenous and are a function of expected future values of themselves and of the predetermined variables. However, since the decision rule is a linear function of the predetermined variables at time t, the forward-looking variables will also be linear functions of these predetermined variables. It may be written as

$$\chi_{t+1} = C_{t+1}X_{t+1}.$$

Then in a rational expectations equilibrium, private sector expectations are formed according to

$$E_t\chi_{t+1} = C_{t+1}E_tX_{t+1} \tag{33}$$

Using (33) to eliminate χ_{t+1} and (22) to eliminate X_{t+1} and χ_t in the right-hand side of (31), the problem can be rewritten in terms of X_t , u_t and V

$$f^{*}(X_t)=X_t^{'}V_tX_t=\underset{u_t}{\mathrm{Min}}\left\{\begin{array}{l}X_t^{'}\tilde{Q}_tX_t+2X_t^{'}\tilde{U}_tu_t+u_t^{'}\tilde{R}_tu_t\\+\beta E_t\left[\left(\tilde{A}_tX_t+\tilde{B}_tu_t+\upsilon_{t+1}\right)^{'}V_{t+1}\left(\tilde{A}_tX_t+\tilde{B}_tu_t+\upsilon_{t+1}\right)\right]\end{array}\right\} \tag{34}$$

where matrices and vectors with a "tilde" are functions of the original Q, U, R, A, B and of C_t. The expression between brackets must be minimised by choosing the vector of instruments, u_t which yields the following first-order conditions:

$$u_t=-\left(\tilde{R}_t+\beta\tilde{B}_t^{'}V_{t+1}B_t\right)^{-1}\left(\tilde{U}_t^{'}+\beta\tilde{B}_t^{'}V_{t+1}\tilde{A}_t\right)X_t \tag{35}$$

or

$$u_t=-F_tX_t \tag{36}$$

Now we can envisage F_t converging to a stationary time-consistent value by iterating on C_t and V_{t+1} which, in the case of convergence, will give the solution under discretion:

- the first n_1 equations can be characterised by

$$X_{t+1}=M^{disc}X_t+\varepsilon_{t+1} \tag{37a}$$

- and the other variables are calculated as

$$u_t=-F^{disc}X_t \tag{37b}$$

$$\chi_t=C^{disc}X_t \tag{37c}$$

Thus, the optimal policy implies setting u_t as a function of only current X_t rather than the sequence $\left\{X_s\right\}_{s=0}^{t}$ as under commitment. Note that discretionary policies are sub-optimal. In forward-looking models, the discretion solution does not take into account the ability of the central bank to influence private sector expectations as is the case in the commitment solution. This creates an inefficiency that results from discretionary policy-making in addition to the familiar inflationary bias.

Finally, it is worth mentioning that if there is no forward-looking variable, there is no difference between discretion and unrestricted commitment: the difference stemming from the shadow prices of the forward-looking variables Ξ_{2t} and the initial conditions associated with them.

5.2.3. *(Restricted) commitment to a simple rule*

The optimal rule under commitment may become quite complex even for simple models in the sense that it may involve many lags of the state variables. Alternatively, the policy maker can commit himself in period t-1 to adhering in period t and forever to a restricted decision rule of the form:

$$u_t = -F\left[X_t^{'}, \chi_t^{'}\right]^{'} \tag{38}$$

In this context, we optimise with respect to the loss function W_0 in (23) and the restriction that the choice of F should give a unique equilibrium in order to obtain the parameters in F. It is a commitment rule, since the policy maker will respect this rule in all subsequent periods, even if it would be optimal to deviate from it in certain cases. This rule is normally not the same as the globally optimal rule (under commitment) since the latter does not restrict the decision rule to be a function of X_t and χ_t (in the simple model y_t and π_t) only. Since there is commitment, the equilibrium is calculated by assuming that the private sector takes (38) for granted when it forms its expectations.

Substitution of (38) into (22) yields

$$\begin{bmatrix} X_{t+1} \\ E_t \chi_{t+1} \end{bmatrix} = (A - BF) \begin{bmatrix} X_t \\ \chi_t \end{bmatrix} + \begin{bmatrix} \upsilon_{t+1} \\ 0 \end{bmatrix} \tag{39}$$

which is a system of first-order difference equations of the same kind as (27) and can thus be solved by applying the same method which yields solutions analogous to (28) and (29). In an equilibrium, predetermined variables can be expressed as function of their own lagged values and the shocks

$$X_{t+1} = MX_t + \upsilon_{t+1} \tag{40}$$

while forward-looking variables are linear functions of predetermined variables

$$\chi_t = CX_t \tag{41}$$

and instruments are of course given by the rule (38). For the simple two-equation model, an analytical solution was presented in section 5-1.

Since the term 'rule' has been used in different ways, it is useful to make the terminology more explicit and to clarify the difference between (17) and (38). In the extensive literature dealing with the definition of a rule, prominent examples are Svensson (1999) or Svensson and Woodford (1999) for a comparison between targeting rules and instrument rules. The following definitions from Svensson are of interest in the present context. A proper *reaction function* expresses the instrument as a function of predetermined variables. If the instrument rule involves forward-looking variables such as (17), it is an equilibrium condition rather than a reaction function. It is called an implicit rule. But it is a simple rule in the sense that it has only a few arguments. In order to find the explicit reaction function expressing the instruments as a function of predetermined variables only, the model (13)-(14) must be solved with the restriction (17) as was done above. A linear *explicit* instrument rule (a linear reaction function) can be written as:

$$u_t = FX_t$$

6. ILLUSTRATIVE EXAMPLES

I now return to the "new synthesis" model in which the monetary authorities set the interest rate in order to minimise the discounted sum of squared deviations of inflation and output gap from their target level. Since the precise empirical performance of the model is not fundamental to the issues examined here, I just give some adhoc values to the parameters of the model. This defines the structure of the following virtual economy:

$$W_0 = E_0 \left\{ \sum_{t=0}^{\infty} 0.99^t L_t \right\}$$

$$L_t = \{(\pi_t - \pi^*)^2 + 0.5(y_t - y^*)^2 + 0(i_t - i_{t-1})^2\}$$

$$y_t = -0.25[i_t - E_t\pi_{t+1}] + E_t y_{t+1} + \varepsilon_{yt}$$

$$\pi_t = 0.99 E_t \pi_{t+1} + 0.5 y_t + \varepsilon_{\pi t}$$

$$\varepsilon_{\pi t+1} = 0.5\varepsilon_{\pi t} + \upsilon_{\pi t+1}$$

$$\varepsilon_{yt+1} = 0.5\varepsilon_{yt} + \upsilon_{yt+1}$$

$$\upsilon_{\pi t} \sim \text{iid}(0,1) \text{ and } \upsilon_{yt} \sim \text{iid}(0,1)$$

When a simple rule is used, coefficients for output and inflation are those proposed by Taylor (1993b),
$F_y = 0.5$;
$F_\pi = 1.5$;
$F_i = 0$ if the central bank does not care about interest rate variability.

Parameters in this simple rule are not derived in order to give the best possible outcome in terms of the welfare function, and the rule is therefore not an "optimal" simple rule. The loss function implies that inflation is twice as important as output and that the central bank does not care about fluctuations in the nominal interest rate. However, if it wants to smooth the interest rate, both λ_i and F_i will be fixed at 0.5.

6.1. New synthesis model : basic model

The economy described by the system of equations (13) and (14) implies that lagged values of any endogenous variables play no role whatsoever in the determination of the equilibrium values of inflation, output or interest rates at a given date. On the other hand, they involve important dynamic links between expectations of the future outcomes to the present state of the economy through the presence of both $E_t y_{t+1}$ and $E_t \pi_{t+1}$ in the equations determining equilibrium at date t. Being purely forward-looking, this model overemphasises the difference between discretion and commitment[7].

Figures 2(a) to (c) show impulse response functions following a cost-push shock $\varepsilon_{\pi 1} = \sigma_\pi = 1$, implying an increase in the inflation rate by one percentage point if there is

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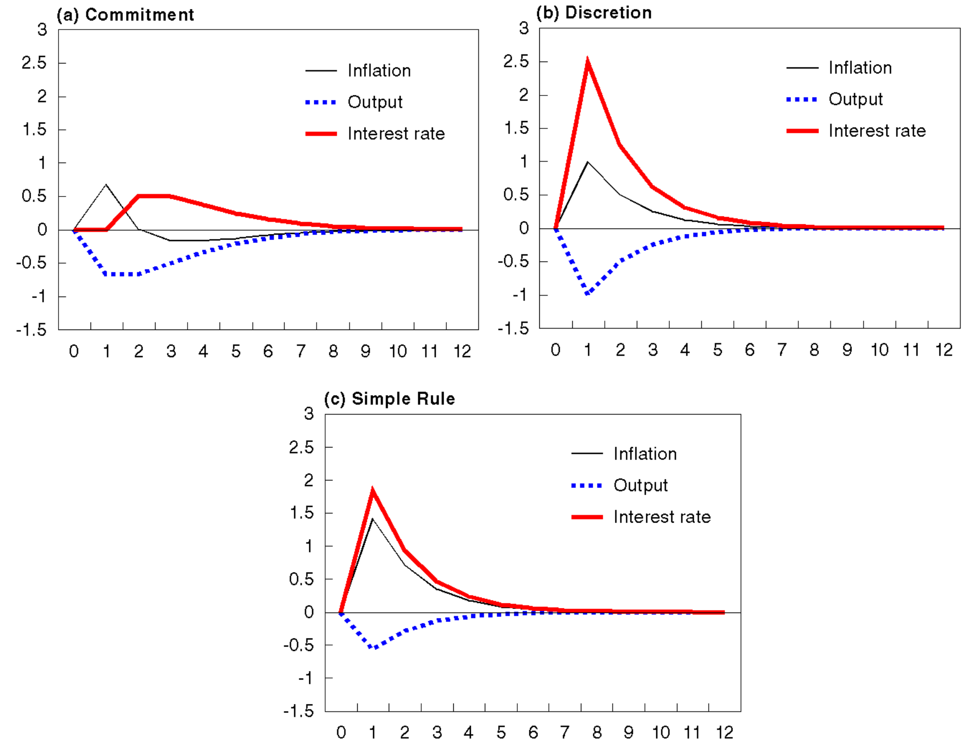
[7] The Matlab codes developed by Söderlind and Klein are used in what follows.

no movement in the output gap or inflation expectations. Equation (20) implies that the shock is autocorrelated and dies out geometrically.

The dynamics of the model stem from the persistence of the shocks according to (20) and (21). However, as shown by comparing equations (29) and (37), the commitment equilibrium contains more dynamics than the equilibrium in the discretion case. Under both optimisation procedures, the cost-push shock increases inflation. The central bank responds by raising the nominal interest rate by more than inflation in order to increase the real rate, although the commitment equilibrium is characterised by a sort of gradual adjustment stemming from its richer dynamics. This, in turn, causes demand to contract below capacity and via the Phillips curve also exerts a negative impact on inflation. However, the records in terms of target variables differ with the optimisation procedure. A central bank which realises that private agents are forward-looking must also realise that the evolution of future output and inflation depends not only on its current decisions on interest rates but also on how private agents expect it to conduct monetary policy in the future. It follows that a more desirable outcome may be achieved if the central bank can make private sector expectations of its future interest rate policy adjust in an appropriate way in response to shocks. The extent of this influence on private agents' expectations depends on the credibility of the central bank announcement that it will behave in a certain way in the future as a result of the shocks that have occurred earlier. Depending on this announcement, the following cases can be distinguished:

Figure 2 - New synthesis Model: impulse responses to a cost-push shock



- discretion: because the output gap depends on the future movement of interest rates, current inflation depends not only on current output gaps but also on expected future output gaps. The policy maker wants to convince the public that he prefers to reduce future output today. But promises made in the past do not constrain current policy. Therefore, the central bank will later have an incentive to renege on its promise of contractionary policy and, instead will promise again to act in the future. As the policy maker is unable to give a credible commitment, rational private agents will not expect large future contractions in demand. Therefore, when the central bank cannot affect private sector expectations about its behaviour at later dates, even when forward-looking elements are present in the model, the optimal policy is a function only of the current state of the economy, and thus consists in reducing current output by increasing the current real interest rate but letting future output revert to zero over time as inflation returns to target;

- conversely, the optimal response under commitment is to continue reducing output as long as inflation remains above target. The credible announcement that output will continue to be reduced as long as necessary in the future has the immediate effect of dampening current inflation. Cost-push shocks therefore generate lower current inflation effects under commitment. Intuitively, the central bank can make credible promises about future policy and thereby affect the expectations of private agents, which in turn affect their behaviour today. This credibility allows the central bank to stabilise output and inflation more effectively. Comparing figures 2-a and 2-b clarifies this point: the cost-push shock increases current inflation by a factor of around 0.7 under commitment as compared to 1 under discretion. Moreover, this result is obtained with a less aggressive increase in the current nominal interest rate[8]. Note finally that here, as opposed to Barro and Gordon's (1983) analysis, the gain from commitment does not stem from the desire of the policy maker to push output above potential, but from the forward-looking nature of inflation, and more specifically from the importance of expectations about future policy.

- the simple rule case looks very much like the discretion case, since both result in a decision rule that is linear only in the current state variables. Only the coefficients are different.

The lack of intrinsic dynamics does not allow the basic model to match the data, and consequently, some extensions have been proposed in empirical analysis. I examine below some simple versions of such extensions with the aim of showing how optimal responses to shocks are explained according to different model specifications. The choice of both the appropriate specification and the parameter values is a matter of empirical investigation which is beyond the scope of this paper.

6.2. Predetermined prices

Small-scale theoretical models often make an assumption of short-term price stickiness. Suppose that prices have to be set just one period in advance, that is π_{t+1} is known already in period t. In this case, the demand equation (13) is unaltered but equation (14) is leaded one period to give:

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[8] The impression one might have from figure **2-a** that the interest rate is not raised in period 1 is a consequence of the

$$\pi_{t+1} = \beta E_t \pi_{t+2} + \kappa E_t y_{t+1} + E_t \varepsilon_{\pi t+1} \tag{42}$$

Figures 3-(a) to (c) show impulse responses of the model with predetermined prices to the same autocorrelated cost-push shock as in the basic model. Prices cannot change at the time of the shock (by assumption), but output and the nominal interest rate can:

- Under commitment, the nominal interest rate is indeed modified at time t=1. Remember that the policy maker has two arguments in his loss function: inflation and output. Since the former is fixed for period t=1, the policy maker will try to stabilise the latter provided this strategy does not affect performance in the following periods. He therefore lowers the nominal rate in order to push down the real rate. This is reinforced by the cost-push shock which impacts on future inflation π_2 and thus on inflation expectations in period 1: $E_1\pi_2$ is high and the ex-ante real rate, $(i_1 - E_1\pi_2)$, is low. Such a lowering of the real interest rate makes consumption (here output) more attractive today (in period 1) than tomorrow (period 2), and this mechanism enables the central bank to keep output totally stable today. The low expected output in period 2 drives down π_2 according to the Phillips curve (42), so that inflation rises in period 2 by far less than the initial shock, actually one third.

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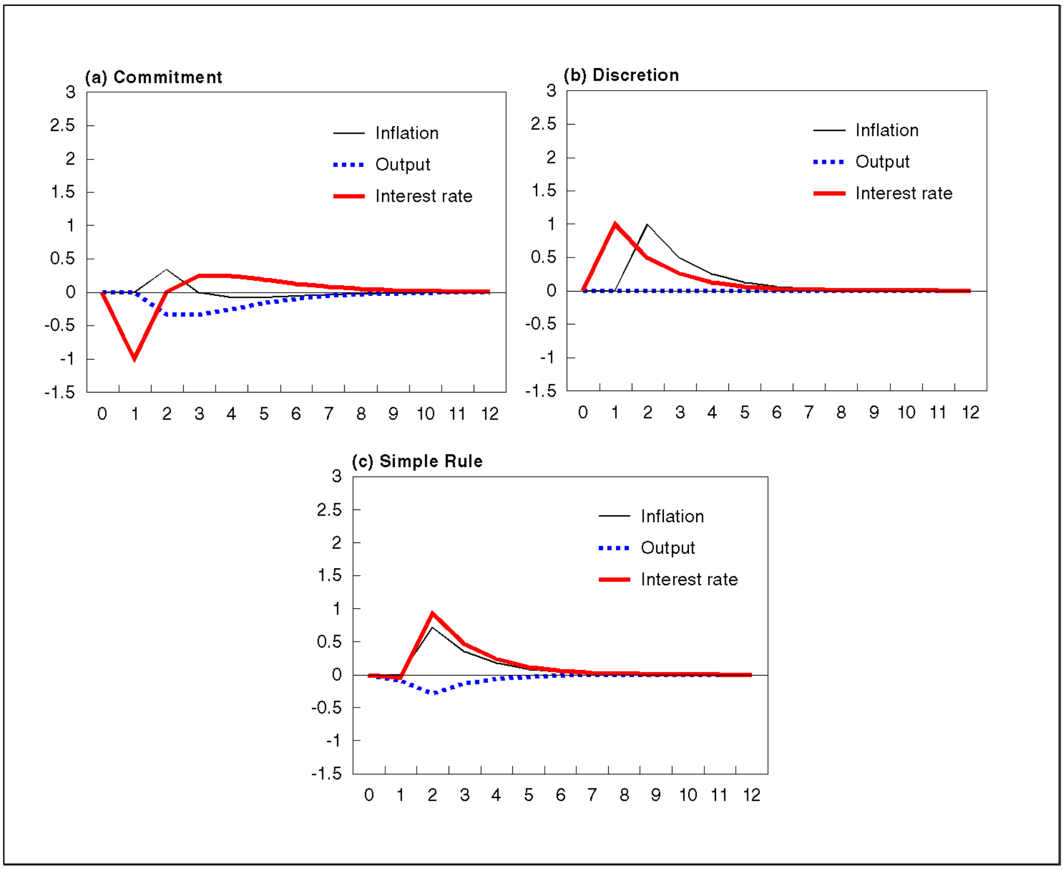
choice of parameter values and is not a general property.

Figure 3 - Predetermined prices: impulse responses to a cost-push shock



- In the simple rule case, the nominal rate also decreases in period 1, but very little, since output does not decrease much in period 1 and the reaction coefficient of the interest rate to the output gap in the policy rule, F_y, equals one half.

- With a discretionary policy, the central bank can keep output totally stable, so that inflation just evolves according to (42) with a zero output gap, i.e. y_{t+1} set to zero, and consequently follows the shock. The relation is slightly different from one-to-one, due to the presence of the discount factor, β. The central bank cannot affect inflation in t by changing i_t since the former was set in period t-1. Thus, being confronted with a price shock, the policy maker who optimises under discretion in period t will stabilise the only variable he can affect in his loss function: output. In order to do this he will set the nominal rate equal to expected inflation in order to leave the real rate unchanged. This is why in figure 3-(b) the interest rate leads inflation by one period.

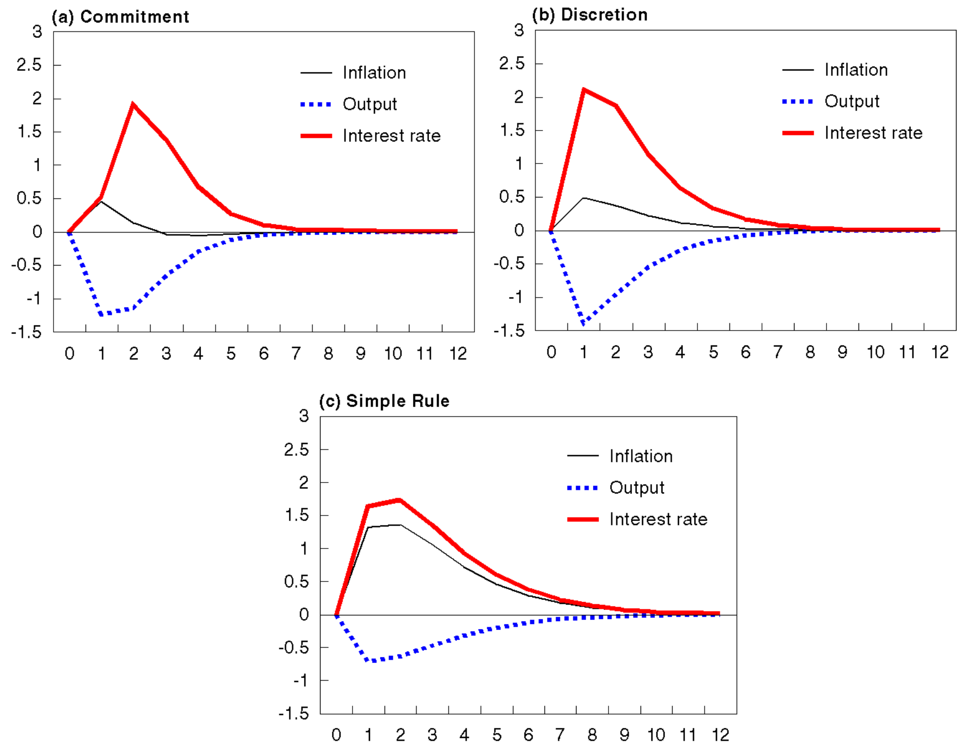
To account for the observed persistence in macroeconomic variables, virtually all major applied macroeconomic models allow for some form of lagged dependence in output and inflation. They will be incorporated in the following sections.

6.3. Inflation persistence : a hybrid Phillips curve

Remember equation (15) which shows that inflation is equal to the present value of future expected marginal costs, implying that inflation is a “jump” variable. Even if prices are set one or two periods in advance, inflation will not exhibit any sluggishness in subsequent periods. This seems at odds with the observed persistence in inflation. Therefore, in order to match the data more closely, several authors have tried to introduce inflation persistence. For instance, Fuhrer and Moore (1995) introduce the hypothesis that agents negociate nominal wage contracts that remain in effect for four quarters. In their contracting decisions, agents compare the current real contract wage with an average of the real contract wages that were negociated in the recent past and those that are expected to be negociated in the near future. This worry about relative wages is used by these authors as a reason for introducing lagged inflation. Svensson (1997) appeals to costs of adjustment or overlapping contracts to justify replacing π_{t+1} by $(1-a)\pi_{t+1}+a\pi_{t-1}$ in his aggregate supply equation. The Gali and Gertler (1999) approach will be followed here. They have suggested the idea that a proportion of firms use a simple rule-of-thumb based on recent history of aggregate price behaviour to set prices. They are referred to as backward-looking firms and their presence allows the introduction of lagged inflation in equation (14). Contrary to Svensson, in this case, the coefficient of $(1-a)\pi_{t+1}+a\pi_{t-1}$ is less than one, implying that there is no vertical Phillips curve in the long run as a consequence of discounting, ($\beta<1$). This is in line with the 'new synthesis' model in which the coefficient of expected inflation is also β. Gali and Gentler call the resulting equation a 'hybrid' Phillips curve.

Figures 4-(a) to (c) give impulse responses to a cost-push shock with a hybrid Phillips curve in which the proportion of backward-looking firms is assumed to be one half. This change does not appear to modify the functioning of the model. Inflation which is assumed to exhibit more sluggishness, increases less on impact but decreases less afterwards as compared to the basic model. Under commitment, this protracted inflationary pressure calls for a tighter monetary policy and also a greater sacrifice ratio which corresponds to the conventional wisdom, stating that the greater the backward-looking part of inflation (e.g. as a consequence of indexation mechanisms), the greater the sacrifice ratio. Under discretion, inflation and the interest rate increase less on impact, but the latter must be maintained at a higher level for a longer period of time. The sacrifice ratio is also larger.

Figure 4 - Hybrid Phillips curve: impulse responses to a cost-push shock

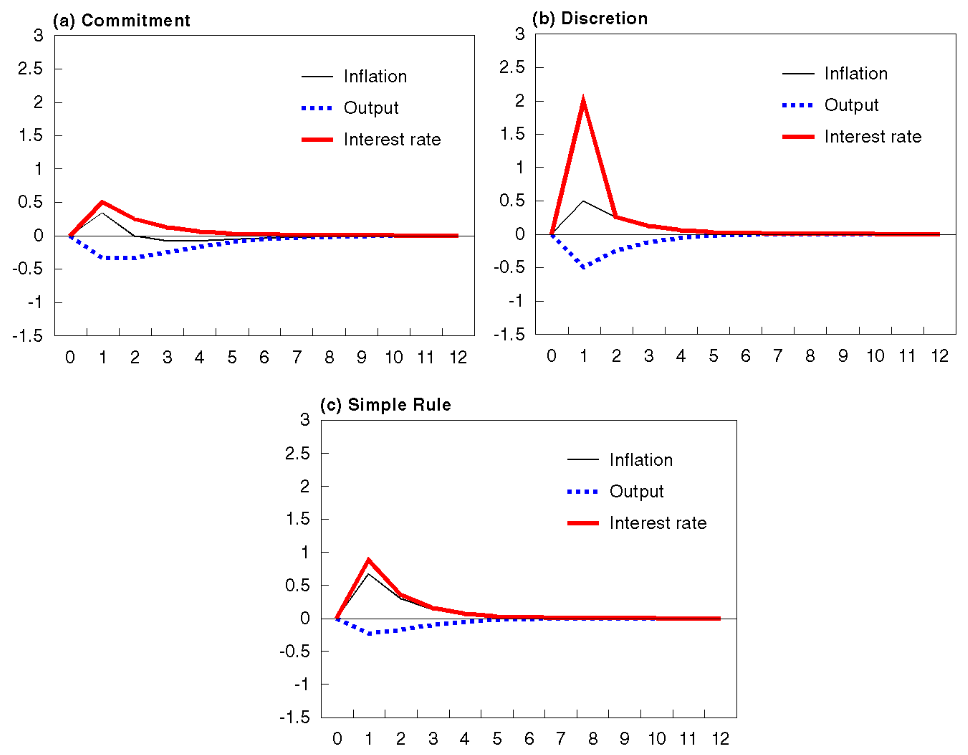


6.4. Output persistence

In the models examined above, a change in monetary policy instantaneously impacted on real output through the contemporaneous relation between the interest rate and output in the IS curve. However, it is well recognised that monetary policy takes time to affect production. Therefore, output persistence has either been introduced in an ad hoc way or has been derived from a utility function that is not additively separable in consumption over time as in McCallum (1999). The latter can be introduced in the simple closed economy used here, while ad hoc specifications are generally used in larger models. Since habit formation makes consumption demand more stable, output is less volatile.

The consequences of the introduction of lagged output in (12) are illustrated in figures 5-(a) to (c). In the commitment case, the nominal interest rate is raised at the time of the shock so that the ex-ante real rate increases much more than in the basic model. Indeed, in order to affect consumption today and to impact on inflation via the aggregate supply curve, the central bank needs to increase the real rate more under habit persistence in order to give agents an incentive to postpone consumption. Thereafter, the interest rate decreases more quickly under all three optimisation procedures.

Figure 5 - Output persistence: impulse responses to a cost-push shock

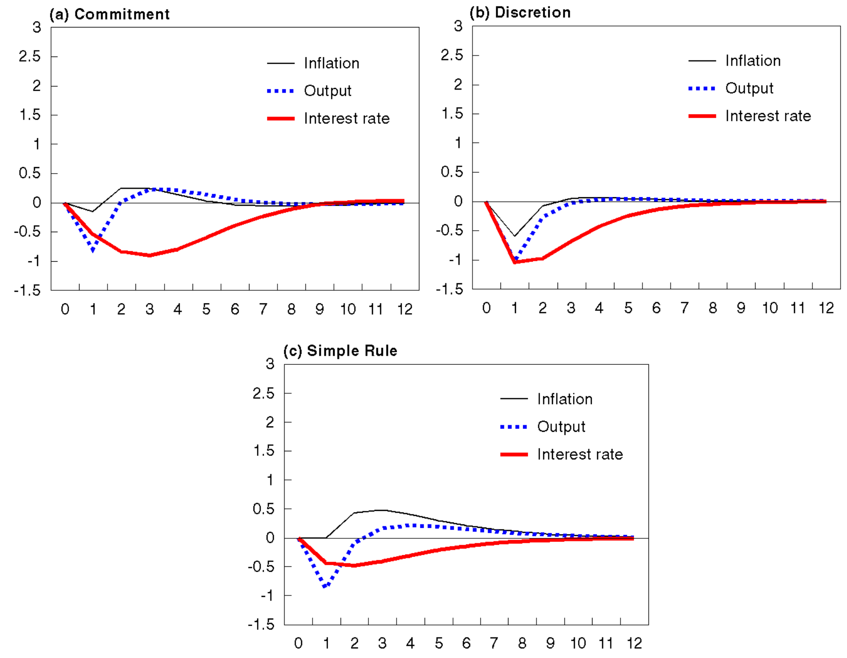


6.5. *Monetary policy inertia*

So far I have only considered alternative structures of the economy. It was supposed that optimal policies did not involve any true element of inertia: any observed persistence in interest rate fluctuations was the consequence of serial correlation in the disturbances and/or lags in the structure of the economy. However, in practice, central banks adjust the interest rate far more cautiously than models predict: optimal policies derived in a certainty-equivalent environment, in which the policy maker knows everything about the world, generally predict much greater volatility in interest rates than is observed. These models lack dynamics in interest rates setting. I therefore now introduce inertia in the central bank's own responses to these disturbances. The desirable degree of inertia in the equilibrium response to shocks can be achieved through an explicit simple rule (a generalisation of the "Taylor rule") in which the interest rate is an increasing function of the lagged interest rate ($F_i>0$), or through the assignment of a loss function that penalises squared interest rate changes ($\lambda_i>0$).

In the absence of any smoothing objective, an optimising central bank sets its interest rates only on the basis of state variables that affect the current or the future determination of output and inflation. Past interest rates are then of no significance in determining the optimal current interest rate level. The problem is now more complicated than the cases considered earlier, since the lagged instrument now enters as a state variable. The discretionary policy is history-dependent because the central bank's loss function is hystory-dependent and the implied reaction function will make the nominal interest rate a function of lagged endogenous variables.

Figure 6 - Monetary policy inertia: impulse responses to a negative demand shock

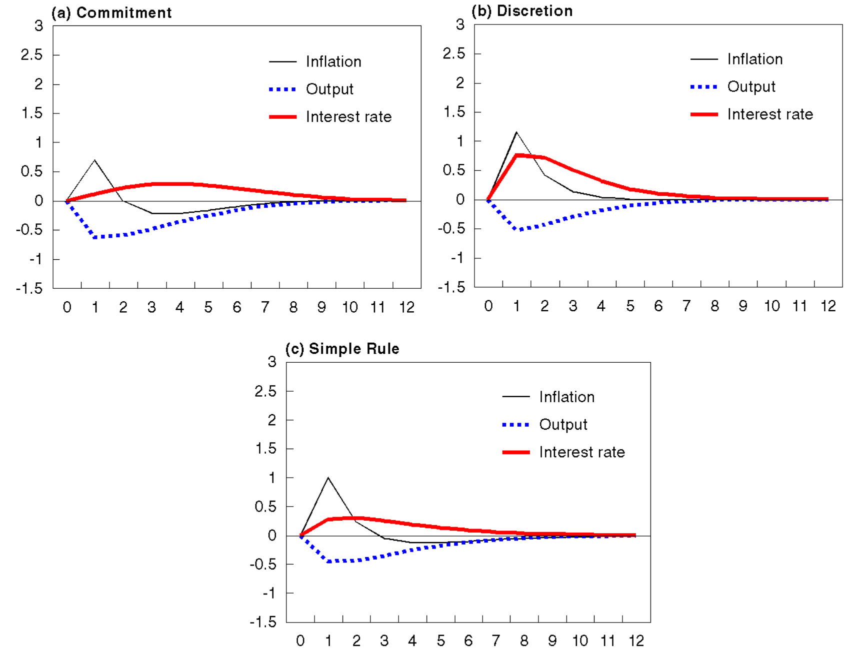


When the central bank does not care about the volatility of the nominal interest rate, $\lambda_i = 0$, then it is always optimal to counterbalance any demand shock in full. This can be seen directly from the aggregate demand curve (13) where any shock ε_{yt} can be compensated by changing i_t by $1/\varphi \ \varepsilon_{yt}$ which according to our parameterisation, would result in a 4 p.c. cut in the short rate to compensate for a one standard deviation negative demand shock. In this way, output is unaffected by the shock. There would then be no effect on inflation either, since the only way the demand shock can affect inflation is via output.

Figures 6-(a) to (c) show impulse responses following a negative demand shock with $\lambda_i = 0.5$. Conversely, in the presence of an inertia motive the interest rate response is smoother and, consequently, output is lower. The way inflation reacts depends on the optimisation procedure. Under optimal commitment and commitment to a simple rule, inflation rises after the first period because people believe that the central bank will maintain the interest rate at a low level to allow output to recover. Under discretion, inflation decreases since future lower rates are not anticipated. It is interesting to note that the introduction of interest rate inertia creates some cyclical pattern in output. The initial negative shock in output is followed by an increase as a consequence of the sustained lower rate, since market agents know that after the initial cut there will be some loss for the central bank to bring the rate back to its initial level. Thus, concerns about interest rate smoothing lead to a less active policy.

In figures 7-(a) to (c), the impulse responses of a cost-push shock are similar to those in the basic model. A price shock implies a temporary increase in inflation which, by causing a monetary policy adjustment, translates into a negative output gap. In comparison with the basic model, interest rate variability is of course reduced. However, the optimal movement in interest rates involves a more persistent increase. Under optimal commitment and commitment to a simple rule, this promise to maintain interest rates at a higher level in the future improves the inflation record with less contraction of output. The optimal response under discretion is now also characterised by a kind of gradual adjustment of the interest rate.

Figure 7 - Monetary policy inertia: impulse responses to a cost-push shock



7. CONCLUSION

Economists are often asked to answer questions about the effects of either exogenous macroeconomic shocks or shifts in economic policies. The need to work with a coherent and well-articulated model of the economy rather than with ad hoc relationships or rules-of-thumb is now well recognised, even for the purpose of forecasting. However, working with such models requires a rather advanced technical background. This paper has examined simple, illustrative versions of more technical and practical models used in the literature with the aim of giving the reader a basic understanding of the treatment of forward-looking expectations.

First, techniques suitable for solving one linear stochastic difference equation were discussed in order to emphasise the need to adequately specify the shocks to be addressed. Thereafter, the paper has focused on the design of optimal monetary policies in small-scale forward-looking models, where inflation is forward-looking and depends on expectations of future inflation and the output gap. It has illustrated that forward-looking variables also complicate the optimisation problem. For example, optimal policy under commitment ceases in general to coincide with the outcome of discretionary optimisation. Although the parameters of these models have been assumed rather than estimated, this analysis has shown step by step how a simple model can be adapted to match the data more closely. For practical economic policy, one can build simple theoretical models that approximate the key features of larger more sophisticated ones. These models can then be used efficiently in forecasting exercises to address specific questions about the future pattern of interest rates and exchange rates, e.g. constant versus endogenous rates, in an informed manner, and also to gauge the actual policy stance.

Important aspects of the research agenda were neglected in this paper, such as the various econometric techniques used to empirically verify these models. Throughout the paper, the certainty-equivalence principle was maintained. Recent research has also dealt with cases where certainty-equivalence does not hold: asymmetric information between the private-sector and the central bank, non-linear models, non-additive parameters or model uncertainty. Another interesting topic in the literature is the problem of partially observable variables, and the process of learning associated to it . All these complications can be classed as extensions, albeit fairly important ones, of the simple tools presented here.

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